

Mathematical foundations for some numerical methods in magnetoencephalography

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1 Introduction

Our purpose is to give mathematical studies for new algorithms applied to MEG. Concerning the direct problem, we follow the formulations of [1-5]; Maxwell's equation controls the total current density $J(x)$ and the magnetic field $B(x)$ in such a way that $\nabla \cdot B = 0$ and $\nabla \times B = \mu_0 J$. We assume that $J = J_p - \sigma(x)\nabla V$, where $J_p(x)$ is the neuron current, and $E = -\nabla V$ stands for the electric field caused by J . The brain in consideration is denoted by a bounded domain $\Omega \subset \mathbf{R}^3$ with boundary $\partial\Omega$. Therefore, $J_p(x)$ is a vector field on $\bar{\Omega}$ with the null normal component on $\partial\Omega$. An important assumption is made on the conductivity $\sigma(x)$; $\sigma(x) = \sigma_1$ ($x \in \Omega$) and $\sigma(x) = \sigma_0$ ($x \in \bar{\Omega}^c$), where $\sigma_{1,0}$ are nonnegative constants. If Ω is a sphere and $\sigma_0 = 0$, then we have a formula of [2] for $J_p(x)$ to determine $B(x)$ directly. In the classical theory, it is assumed that the primary current $J_p(x)$ is a combination of dipoles such as

$$J_p(x) = \sum_{k=1}^N Q_k \delta(x - a_k).$$

Unknown parameters $\{a_k, Q_k\}$ are determined to adjust with discretely measured values of $\{n \cdot B\}$ on $\partial\Omega$ in use of the method of least square approximation (LSA). One aspect of the *method of current element distribution* is a trial to recover the total current density $J_p(x)$. First observation is that if $J_p(x)$ is a single dipole $Q_k \delta(x - a_k)$, then it produces the magnetic field

$$\frac{\mu_0}{4\pi} \frac{Q_k \times (x - a_k)}{|x - a_k|^3}.$$

Let the number of channels, their positions, and the directions be M , $x_j \in \partial\Omega$, and $v_j \in \mathbf{S}^2$, respectively, where $1 \leq j \leq M$ (usually, around 100 observing points are taken). Taking N sufficiently large (around 200), one introduces the mapping

$$\varphi : (Q, a) \in \mathbf{R}^{6N} \rightarrow$$

$$\left(\frac{\mu_0}{4\pi} \sum_{k=1}^N \frac{Q_k \times (x_j - a_k)}{|x_j - a_k|^3} \cdot v_j \right)_{1 \leq j \leq M} \in \mathbf{R}^M$$

and the functional

$$\vartheta(Q, a) = \frac{1}{2} \|\varphi(Q, a) - z\|_{\mathbf{R}^M}^2,$$

where

$$Q = (Q_1, Q_2, \dots, Q_N) \in \mathbf{R}^{3N},$$

$$a = (a_1, a_2, \dots, a_N) \in \mathbf{R}^{3N}$$

are parameters to be determined, and

$$z = (z_1, z_2, \dots, z_M) \in \mathbf{R}^M$$

with $z_j \in \mathbf{R}$ standing for the observed datum of the v_j component of the magnetic field $B(x)$ measured at the j th channel. Each $Q_k \delta(x - a_k)$ is called the *current element*. First, N random current elements are distributed on $\bar{\Omega}$. One element is selected randomly and perturbed so that ϑ decreases. Then the next one is chosen also randomly to continue the process. If ϑ does not decrease, that element is left un-perturbed. If the directions of two adjacent elements happen to be opposite, then randomly selected one of them is moved away. This procedure is continued until a local minimum of ϑ is achieved.

2 Second optimization

Serious discussions have to be made on the relation between the number of assumed dipoles and that of channels; its balance leads LSA to be either under-determined or over-determined, of which difficulties are essentially different.

Given $\varphi : \mathbf{R}^n \rightarrow \mathbf{R}^m$ and $z \in \mathbf{R}^m$, the discrete inverse problem is formulated as to find $x \in \mathbf{R}^n$ satisfying $\varphi(x) = z$, of which solution is called the *strict solution*. It is over-determined and under-determined according to $n < m$ and $n > m$, respectively. In the over-determined case, the strict solution is hard to obtain; the discrete inverse problem is reformulated as the least square problem of finding $x \in F$ satisfying $\vartheta(x) = j$, where

$$\vartheta(x) = \frac{1}{2} \|\varphi(x) - z\|^2 \quad \text{and} \quad j = \inf_F \vartheta.$$

Its solution is called the *least square solution*, but practically one can examine its local minimality only. If $\vartheta(x)$ is satisfactorily small, one says that x has a *high accuracy*. Any local minimum with high accuracy is regarded as a least square solution. Sometimes it is not unique. One has to select a particular solution, regarding a priori information and their accuracies. The least square problem is said to be *quasi-identifiable* if any local minimum of ϑ with high accuracy is isolated. This allows one to select one of them as a desirable solution. A criterion for quasi-identifiability, given by [6] is the high-accuracy and the rank condition. The latter means that $\varphi'(x_0)^* \varphi'(x_0) : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is non-singular. Because the rank condition is generically holds by $n < m$, only high-accuracy is to be examined in that case.

However, the current element distribution method is set to be under-determined. Generically, the set of quasi-solutions $M = \{x \mid \varphi(x) = z\}$ forms a manifold of $n - m$ dimension. The iterative sequence $\{x_l\}$ approaches M . Then it *freezes* in the region of $\|\Delta x_{l+1}\|^2 \sim \vartheta(x)$ for $\Delta x_{l+1} = x_{l+1} - x_l$, because in this range the accuracy is hard to improve. Making use of the tangent space $T_{x_l} M$ as the perturbation set, one can remove the freezed sequence without losing the high-accuracy. This process, called *melting*, is realized by the singular decomposition of the matrix $\varphi'(x_l)$, because $\vartheta(x_l) \ll 1$ implies $T_{x_l} M \approx \text{Ker} \varphi'(x_l)$.

In the actual process of MEG, the above processes are realized in the following way. First, in approaching, mainly $Q \in \mathbf{R}^{3N}$ is perturbed. Also, the maximum descendent method is better than the random perturbation here. On the other hand, in melting, mostly $a \in \mathbf{R}^{3N}$ is moved. Furthermore, current elements are so taken to be clustered. We call this special process of melting the *binding*. One method for binding is as follows. First, for $\varepsilon > 0$ small, we take an ε covering of $S = \{a_k \in \mathbf{R}^3 \mid 1 \leq k \leq N\}$. Fixing $\varepsilon > 0$, this process is repeated in use of the random perturbation until the least number of the covering balls is obtained. We call it *covering*. Then, those balls are contracted as small as possible, remaining to be a covering Σ . We call this process the *biting*. After then, the melting is done under the constraint that any element does not move away from those contracted balls. Taking smaller $\varepsilon > 0$, we repeat the process.

For computations indicated by the next figures, a more rough method is adopted. There, only one ball is taken to contain Σ . Based on this covering, biting and melting are done similarly. Observing points are 16, and current elements are 30. Then it was seen that current elements are clustering near the originally clustered 8 dipoles.

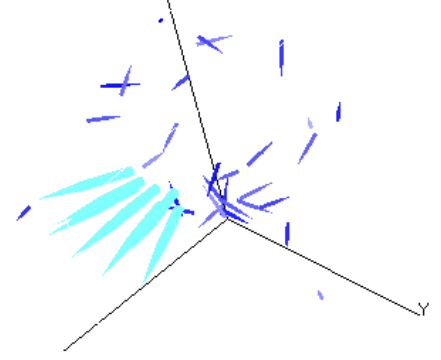


Figure 1: Initial State

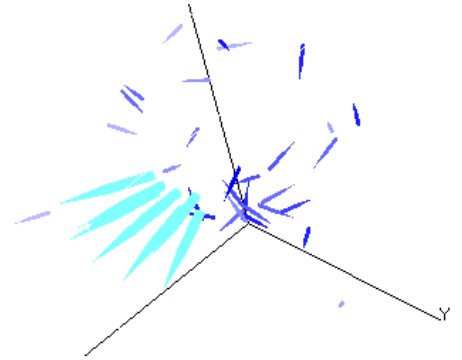


Figure 2: Freezing State

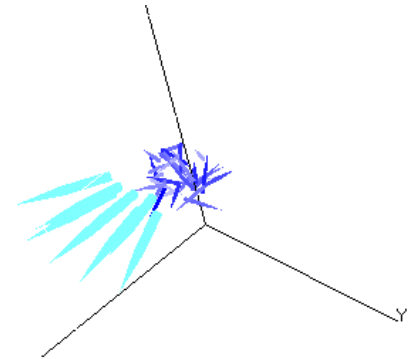


Figure 3: After 800 Times Melting

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